Involutive Spacetime Distributions
and
p-Brane Dynamics

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Abstract

We propose a precise definition of multidimensional fluids generated by self-gravitating extended objects such as strings and membranes: a p-dimensional perfect fluid is a smooth involutive p-dimensional distribution on a spacetime, each integral manifold of which is a timelike, connected, immersed submanifold of dimension, p – representing the history of a $(p-1)$-dimensional extended object. This geometric formulation of perfect fluids of higher dimensions naturally leads to the associated stress-energy tensor. Furthermore, the laws of temporal evolution and symmetries of such systems are derived, in general, from the Einstein field equations and the integrability conditions. We also present a matter model based on a 2-dimensional involutive distribution, and it is shown that the stress-energy tensor for self-gravitating strings gives rise to a non-trivial spherically symmetric spacetime with a naked singularity.

1 Introduction

The purpose of this work is to develop a general relativistic theory of multidimensional fluids as sources of spacetime curvature. The basic ingredients of such a fluid are $(p-1)$-dimensional spatially extended objects called p-branes – where $p = 1, 2, 3$ correspond to point particles, strings and membranes respectively. More precisely, a p-brane is a timelike, connected, p-dimensional C^∞-manifold immersed in a spacetime – representing the temporal evolution of a $(p-1)$-dimensional extended object. Then, for a fixed p, a multidimensional fluid is defined by a smooth involutive p-dimensional distribution on a spacetime, each integral manifold of which is a p-brane. Thus, a multidimensional fluid naturally generalizes the model of a collisionless gas of point particles by a congruence of world lines (1-dimensional distribution). [In this paper, all manifolds, tensor fields on them, and all maps from one manifold to another will be C^∞. Also, we define a spacetime as a non-compact, connected, oriented and time-oriented n-dimensional manifold, M, endowed with a Lorentz metric, g. For $n = 4$, the corresponding spacetime will be denoted (M^4, g).]

In a class of field theoretic models[1−4] of the early universe, p-branes appear as 'topological defects' with characteristic rest-mass per unit $(p-1)$-dimensional spatial volume. The existence of such extended objects could be a possible source of density perturbations, and hence may provide a causal mechanism for generating the observed large scale structure of the universe[5]. Thus any scheme, based on general relativity, for investigating the role of multidimensional fluids in the evolution of the universe, requires specification of a stress tensor, solutions of Einstein's field equations, description of the behaviour of other forms of matter in the vicinity of the extended objects, and characterization of the resulting spacetimes and their singularities.

In order to carry out this program, we need a precise form of the fluid stress tensor, T, which is formally a symmetric $(0, 2)$-tensor field on a spacetime (M, g). Physically T replaces and unifies the concepts of energy density, momentum density, energy flux and momentum flux. These quantities are observer dependent. [An *observer* is a future-pointing timelike curve $\gamma : I \longrightarrow M$ ($I \subset \mathbf{R}$ is an open interval) such that $\forall s \in I$, the tangent vector $\gamma_{*s} \in T_{\gamma(s)}M$ satisfies $g(\gamma_{*s}, \gamma_{*s}) = -1$. An *instantaneous observer* (x, Z) at $x \in M$, is a future-pointing timelike unit vector $Z \in T_x M$.] When an instantaneous observer (x, Z) in (M^4, g) measures, for instance, the energy density in any unit 3-volume of the local rest space $Z^\perp \equiv \{X \in T_x M^4 | g(X, Z) = 0\}$, $\mathsf{T}(Z, Z)$ corresponds to the measured energy density.

Also, for all known forms of matter $\mathsf{T}(Z, Z) \geq 0$, $\forall$ instantaneous observer (x, Z) (and hence by continuity $\mathsf{T}(X, X) \geq 0$, $\forall$ causal $X \in T_x M^4$) $\forall x \in M^4$. This operational definition uniquely specifies T in the following sense[6]:

<u>Theorem</u>: If the symmetric $(0,2)$ tensor fields T and T' on a spacetime (M, g) satisfy $\mathsf{T}(Z, Z) = \mathsf{T}'(Z, Z)$ for all instantaneous observers (x, Z) then $\mathsf{T} = \mathsf{T}'$.

Hence measured energy density $\mathsf{T}(Z, Z)$ naturally motivates the following

<u>Definition 1</u> : A stress-energy tensor on spacetime M is a symmetric $(0, 2)$-tensor field T on M such that $\mathsf{T}(X, X) \geq 0$ for all causal $X \in T_x M$, $\forall x \in M$.

Based on the Theorem above, we shall motivate the definition, (1.3), of the stress tensor for a collisionless gas of point particles (of mass m) in a way that is suitable for generalization to multidimensional fluids. Such a fluid on (M^4, g) is a congruence of integral curves of a nowhere vanishing (energy-momentum) vector field, P, on a spacetime region – where $g(\mathsf{P}, \mathsf{P}) = -m^2$. This configuration of (integral) curves is a 1-dimensional involutive distribution on M^4 (or 1-foliation of M^4).

The above geometric structure of a particle flow suggests that a collection of non-colliding extended particles (non-intersecting p-branes) in a spacetime, (M^n, g), can be modelled by a p-dimensional $(p \geq 1)$ foliation of (M^n, g) – where the timelike integral manifolds represent the p-branes. After a brief introduction to foliations in section 2, we then characterise a multidimensional fluid, in section 3, as a p-foliation determined locally by a *nowhere zero* decomposable p-form $\omega = \sigma(\tilde{V}_1 \wedge \cdots \wedge \tilde{V}_p)/\mu$ where $\{\tilde{V}_a\}_{a=1}^p$ are metric dual of local vector fields $\{V_a\}$ giving (local) bases for tangent spaces of integral manifolds of the p-foliation, $\det[g(V_a, V_b)] = -\mu^2 \neq 0$, and σ is the characteristic rest mass per unit $(p-1)$-dimensional spatial volume of the p-branes satisfying $\mathcal{G}(\omega, \omega) = -\sigma^2 \neq 0$. Here, $\mathcal{G}$, is a scalar product on the vector space of differential p-forms, $\Lambda^p(M^n)$ [see Appendix].

This local foliation p-form, ω, together with a density function η [(3.1)] allows us to specify, locally, in a smooth way the number of p-branes of the fluid system in spacelike sections of a spacetime. Such a description of a multidimensional fluid in terms of (ω, η) naturally leads to the associated stress tensor [(3.15)] with the following local representation:

$$\mathsf{T} = \eta \mathcal{G}(\iota_a \omega, \iota_b \omega) e^a \otimes e^b$$

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where, ι_a, is the interior contraction operator on differential forms with respect to any local basis vector fields $\{X_a\}$ with the corresponding dual basis $\{e^a\}$.

In **section 4** we consider such a stress tensor as a possible source of spacetime curvature, and derive its dynamical consequences from the Einstein field equations. In particular we have shown that the foliation p-form ω satisfies a 'conservation law' [**Proposition 4**] :

$$\delta(\eta\omega) = 0$$

and each integral submanifold (p-brane) determined by ω has vanishing mean curvature, $\mathsf{H} = 0$ [**Proposition 3**]. It is also shown that if the spacetime admits a Killing vector field K then the world density, η as well as ω are invariant with respect to the local isometry generated by K [**Proposition 1** and **Proposition 2**] :

$$\mathsf{L_K}\eta = 0 \quad ; \quad \mathsf{L_K}\omega = 0$$

[Here δ and L are coderivative and Lie-derivative operators respectively].

These properties can be used to solve for the foliation p-form, ω, and the world density, η, which specify the multidimensional fluid as well as the spacetime metric. In **section 5**, we have considered a matter model based on a 2-foliation where self-gravitating extended particles do indeed give rise to a non-trivial spacetime with a naked singularity. A new class of gravitational collapse problems is also presented.

1-Dimensional Perfect Fluids

In this subsection we motivate the definition of the stress tensor associated with an 1-dimensional perfect fluid (flow of point particles) on (M^4, g), from the viewpoint of the uniqueness **Theorem** stated above. For $m \in [0, \infty)$, a particle of mass m is a future-pointing curve $\gamma : I \longrightarrow M^4$ such that $g(\gamma_{*s}, \gamma_{*s}) = -m^2$, $\forall s \in I$. Here $m \neq 0$ is the analogue of Newtonian inertial mass and $m = 0$ is allowed. The vector field, γ_*, over γ is called energy-momentum of the particle. Then for an instantaneous observer $(\gamma(s), Z)$, we have the orthogonal decomposition of $\gamma_{*s} \in T_{\gamma(s)}M^4$:

$$\gamma_{*s} = \mathbf{e}Z + \mathbf{p} \tag{1.1}$$

where $\mathbf{e} = -g(\gamma_*, Z) > 0$ is the energy and $\mathbf{p} \in Z^\perp$ is the momentum of the particle as measured by (x, Z), and hence the Newtonian velocity of a particle with respect to Z is

given by $\mathbf{v} = \mathbf{p}/\mathbf{e} \in Z^{\perp}$. Now, if we have enormous number of particles, each having the same mass $m \in [0, \infty)$ and the energy-momenta, then we may describe such a system on M^4 by the following

<u>Definition 2</u> : An 1-dimensional perfect fluid (P, η, m) on (M^4, g) consists of a function η : $M^4 \to [0, \infty)$ called *world density* and an energy-momentum vector field $\mathsf{P} : M^4 \to TM^4$ such that each integral curve of P is a particle of mass m, and the integral of the number density 3-form

$$\mathbf{n} = \star(\eta \widetilde{\mathsf{P}}) \tag{1.2}$$

over a spacelike section $\mathcal{D}^3 \subset M^4$ defines the total number of particles in $\mathcal{D}^3$. Associated with (P, η, m) is the stress-energy tensor

$$\mathsf{T} = \eta \widetilde{\mathsf{P}} \otimes \widetilde{\mathsf{P}} \tag{1.3}$$

[Here $\star$ is the Hodge operator induced by the metric on M^4, and $\widetilde{\mathsf{P}}$ is metric dual of P.]

<u>Remark</u>: It follows from Definition 2 and the definition of a particle (of mass m) that P is future-pointing and $g(\mathsf{P}, \mathsf{P}) = -m^2$. <u>Motivation for (1.3)</u> : T is symmetric, smooth and for all $X \in T_x M^4$, $\mathsf{T}_x(X, X) = \eta_x[g(\mathsf{P}, X)]^2 \geq 0$. Thus T is a stress-energy tensor, by Definition 1. We now explain in what sense the measured energy density is $\mathsf{T}_x(Z, Z)$ for every instantaneous observer $(x, Z) \in T_x M^4$.

Given an observer (x, Z), (1.2), by regarding part of $T_x M^4$ (using exponential map) as a part of M^4 for a sufficiently small neighbourhood of $x \in M^4$, where curvature tensor is negligible[6]. Then, given a set of linearly independent vectors X_1, X_2, X_3 $\in Z^{\perp}$ (rest space of Z), the world density function η can be interpreted as follows : the number of particles measured by (x, Z) in the parallelopiped $[X_1 X_2 X_3] \subset Z^{\perp} \subset T_x M^4$ is the number of integral cuves of P crossing the parallelopiped $[X_1 X_2 X_3]$, and is (approximately) given by

$$\begin{aligned} |\mathbf{n}(X_1, \ X_2, \ X_3)| \ &= \ \eta_x |\Omega(\mathsf{P}, \ X_1, \ X_2, \ X_3)| \\ &= \ \eta_x \mathbf{e} |\Omega(Z, \ X_1, \ X_2, \ X_3)| \end{aligned} \tag{1.4}$$

where $\Omega \equiv \star 1$ is the volume form on M^4, $|\Omega(Z, \ X_1, \ X_2, \ X_3)|$ is the 3-volume of the parallelopiped $[X_1 X_2 X_3]$, and from the Definition 2 and (1.1), P is given by the following orthogonal decomposition

$$\mathsf{P}_x = \mathbf{e} Z + \mathbf{p} \tag{1.5}$$

An alternative way to calculate the particle number density in $Z^\perp$ is to project, $\mathbf{n}_x$ [(1.2)], into $\Lambda^3(Z^\perp)$ – the vector space of 3-forms on $Z^\perp$ – by the $\mathbf{R}$-linear map, $\Pi_Z : \Lambda^3(T_x M^4) \longrightarrow \Lambda^3(Z^\perp)$, where $\Pi_Z = 1 + \widetilde{Z} \wedge \iota_Z$. By (1.5),

$$\mathbf{n}_Z \equiv \Pi_Z[\star(\eta\widetilde{\mathsf{P}})]_x = \eta_x \mathbf{e} \star \widetilde{Z} \tag{1.6}$$

Then the number of particles (x, Z) measures in any unit volume of the local rest space $Z^\perp$, is given by

$$\|\mathbf{n}_Z\| \equiv [\mathcal{G}(\mathbf{n}_Z, \mathbf{n}_Z)]^{\frac{1}{2}} = \eta_x \mathbf{e} \tag{1.7}$$

where $\mathcal{G}$ [see Appendix] is the non-degenerate symmetric bilinear form, induced by g, on the vector space $\Lambda^p(M^4)$ of differential p-forms. From (1.5) and (1.7) we compute the energy density U measured by (x, Z) :

$$U \equiv (\eta_x \mathbf{e})\mathbf{e} \tag{1.8}$$

Thus T is a stress tensor for the particle flow (P, η, m), and by (1.3), (1.5), (1.8) we have

$$\mathsf{T}_x(Z, Z) = \eta_x[g(\mathsf{P}, \ Z)]^2 = \eta_x \mathbf{e}^2 = U \tag{1.9}$$

for every instantaneous observer Z. Hence $\mathsf{T}_x(Z, \ Z)$ is the energy density of the particle flow (P, η, m) – measured by (x, Z). Then, by the uniqueness property, the stress-energy tensor for a particle flow is indeed specified by (1.8).

2 Involutive Distributions (or Foliations)

In order to generalize the notion of particle flows, we recall the following

<u>Definition 3</u> : A p-dimensional smooth distribution D on a manifold M^n is an assignment, to each point $x \in M^n$, of a p-dimensional subspace D_x of $T_x M^n$.

Remarks: The smoothness of D can be expressed in two equivalent ways :

(1) Every $x \in M^n$ has a neighbourhood $\mathcal{U}_x \subset M^n$ on which there exists a set of smooth (local) vector fields $\{V_\alpha : \alpha = 1, \ldots, p\}$ such that the vectors $(V_\alpha)_y$ is a basis for the subspace distinguished by the distribution D_y for every $y \in \mathcal{U}_x \subset M^n$. Thus $\{V_\alpha\}$ are said to span the distribution, locally.

(2) Every $x \in M^n$ has a neighbourhood $\mathcal{U}_x \subset M^n$ on which there exist $(n-p)$ independent smooth (local) 1-forms $\{\theta^k : k = p+1, \ldots, n\}$ such that $\theta^k|_{\mathsf{D}_y} = 0$ for all $y \in \mathcal{U}_x \subset M^n$. Thus $\{\theta^k\}$ are called constraint 1-forms for D. If D is locally spanned by $\{V_\alpha\}$, then $\theta^k(V_\alpha) = 0$.

Now, an immersed submanifold $\mathcal{S}^p$ in M^n is said to be an integral manifold (also called a 'leaf') of D, if at each point $x \in \mathcal{S}^p$, its tangent space $T_x\mathcal{S}^p$ coincides with the subspace D_x of T_xM^n. Then a distribution D is called integrable (or involutive) if through each point of M^n there is an integral manifold of D, and the necessary and sufficient condition (Frobenius's integrability condition) for D to be integrable is given by[8,9]

$$[V_\alpha, \ V_\beta] = f^\gamma_{\alpha\beta} V_\gamma \tag{2.1}$$

for some local functions $f^\gamma_{\alpha\beta}$ on M^n. Or equivalently

$$d\theta^k = \lambda^k{}_j \wedge \theta^j \tag{2.2}$$

for some local 1-forms $\lambda^k{}_j$ on M^n. An integrable distribution is called a **foliation**.

3 p-Dimensional Perfect Fluids

We recall from **section 1** that a p-brane is defined by a timelike, connected, p-dimensional manifold immersed in a spacetime and is distinguished by a strictly positive parameter, σ - the rest mass per unit $(p-1)$-dimensional spatial volume. Thus, we introduce the following

<u>Definition 4</u> : A p-dimensional perfect fluid, $(\mathsf{D}, \eta, \sigma)$, in a spacetime (M^n, g) consists of a function $\eta : M \to [0, \infty)$, called world density, and a smooth integrable p-dimensional distribution D on M such that each integral manifold of D is a p-brane of rest mass per unit spatial volume, σ. If the independent local vector fields $\{V_1, \ldots, V_p\}$ span D on the open set $\mathcal{U} \subset M^n$ and $\omega \equiv \sigma(\tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p)/\mu$ where $\det[g(V_\alpha, V_\beta)] = -\mu^2 \neq 0$, so that $\mathcal{G}(\omega, \omega) = -\sigma^2$, then the **integral** of the **local number density** $(n-p)$-**form**

$$\mathbf{n} = \star(\eta\omega) \tag{3.1}$$

over a **spacelike** $(n-p)$-**chain** $\mathcal{C} \subset \mathcal{U}$, defines the total number of p-branes in $\mathcal{C}$.

Remarks:

(1) <u>Locally,</u> D is spanned by a set of vector fields $\{V_\alpha \ : \ \alpha = 1, \ldots, p\}$, which forms a basis for each timelike tangent space of each integral manifold (p-brane) of D. Then p-branes in $(\mathsf{D}, \eta, \sigma)$ are **locally** characterised by a decomposable p-form on M^n, $\chi \equiv \tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p$, with

$$\omega \equiv \sigma(\tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p)/\mu \tag{3.2}$$

$$\omega^{(p)} \equiv (\tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p)/\mu \tag{3.3}$$

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$$\mathcal{G}(\chi, \chi) = \det[g(V_\alpha, V_\beta)] = -\mu^2 \tag{3.4}$$

where μ is a strictly positive, real-valued, local function. The negative sign in (3.4) reflects the timelike causal character of each integral manifold of D.

(2) In (3.3), the p-form $\omega^{(p)}$ (hence, ω and $\mathbf{n}$) is independent of the choice of the vector fields $\{V_\alpha\}$ that span D, locally. Moreover, when restricted to a p-brane, $\omega^{(p)}$ is the induced volume form on the corresponding timelike integral manifold with $\mathcal{G}(\omega_p, \omega_p) = -1$.

(3) A p-brane of rest mass per unit spatial volume, σ, is a timelike immersion with local parametrisation $\phi : (0,1)^p \to M$ such that for $\phi(s^1, \ldots, s^p) = x \in M^n$, $\phi_\star(\partial_{s^\alpha}) = v_\alpha \in T_x M^n$ and the p-form $\omega_0 \equiv \sigma(\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_p)/\mu_0$ satisfies

$$\mathcal{G}(\omega_0, \omega_0) = -\sigma^2 \neq 0 \tag{3.5}$$

where $\det[g(v_\alpha, v_\beta)] = -\mu_0^2$, and σ is the Newtonian analogue of inertial energy per unit spatial volume of the p-brane[10]. We also recall that in a curved spacetime, a p-brane is self-gravitating if ϕ is an extremal immersion (and hence, the mean curvature of ϕ vanishes).

Thus given a foliation of a spacetime (M, g) determined by a distribution D, the corresponding multidimensioal fluid, (D, η, σ), is locally characterised by the decomposable p-form ω on M. In order to motivate the definition, (3.15), of a stress-energy tensor T for such systems, we now give an approximate local analysis to obtain the energy density U_Z with respect to any instantaneous observer (x, Z) for all $x \in M$.

Energy Density U_Z for (D, η, σ) :

For any $x \in M^n$, let $\mathcal{U}_x$ be the neighborhood of x where (D, η, σ) is locally represented by (ω, η), and consider a p-brane through x. Then, given an instantaneous observer $(x, Z) \in T_x M^n$, ω admits the following orthogonal decomposition with respect to (x, Z):

$$\omega_x = \tilde{Z} \wedge (-\iota_Z \omega_x) + \Pi_Z \omega_x \tag{3.6}$$

where $\Pi_Z \equiv 1 + \tilde{Z} \wedge \iota_Z$ is the projection operator, $\Pi_Z \omega_x$ is supported on (x, Z)'s rest-space, $Z^\perp \subset T_x M^n$ in the sense that $[\Pi_Z \omega_x](Z) = 0$, and $\mathcal{G}(\tilde{Z} \wedge (-\iota_Z \omega_x), \Pi_Z \omega_x) = 0$. Now, for a p-brane through x, its energy per unit spatial volume, $\mathcal{E}_Z$, with respect to (x, Z) is defined by

$$\mathcal{E}_Z = [\mathcal{G}(\iota_Z \omega_x, \iota_Z \omega_x)]^{\frac{1}{2}} \tag{3.7}$$

To motivate this definition, we look at the relevant properties of $(p-1)$-form $\Theta \equiv \iota_Z \omega_x = (\sigma/\mu_x)\iota_Z \chi_x$.

(a) From the equations (3.2) and (3.4) we have

$$\Theta = (\sigma/\mu_x) \sum_{\alpha=1}^{p} (-1)^{\alpha-1} g(V_{\alpha x}, Z) \chi_x^\alpha \tag{3.8}$$

where the $(p-1)$-form χ^α is defined by

$$\chi^\alpha \equiv \tilde{V}_1 \wedge \ldots \wedge \tilde{V}_{\alpha-1} \wedge \tilde{V}_{\alpha+1} \wedge \ldots \wedge \tilde{V}_p \tag{3.9}$$

Note that one of the local vector fields, $\{V_\alpha\}$, say V_1, must be causal since the set $(V_\alpha)_x$ forms a basis for a timelike subspace of $T_x M$. Then, $g(V_1, Z) \neq 0$ since Z is timelike, and hence by (3.8) $\Theta \neq 0$. Since, $\iota_Z \Theta = \omega_x(Z, Z) = 0$ and ω_x is decomposable, the dimension of the characteristic subspace[8] of the non-zero $(p-1)$-form Θ is $(n-p+1)$, and hence Θ is also decomposable and can be written as

$$\Theta = (\sigma/\mu_x)(\tilde{Y}_1 \wedge \ldots \wedge \tilde{Y}_{p-1}) \tag{3.10}$$

where $\{Y_1, \ldots, Y_{p-1}\}$ are linearly independent vectors in $T_x M$. Now, $\iota_Z \Theta = 0$ and the linear independence of the Y_a's imply

$$g(Y_a, Z) = 0 \quad \forall a = 1, \ldots, p-1 \tag{3.11}$$

From (3.11) it follows $Y_a \in Z^\perp$, and hence each Y_a is spacelike and $\mathcal{G}(\Theta, \Theta) > 0$.

(b) For any normal field N where $g(N, V_\alpha) = 0$ for all α, we find $\iota_N \Theta = \iota_N \iota_Z \omega = 0$. Then by (3.10) it follows that $g(Y_a, N) = 0$ and hence each spacelike Y_a also belongs to $\mathcal{V}_x^p$ - the tangent space of a p-brane through $x \in M$, which is spanned by the set of independent vectors $\{V_{1x}, \ldots, V_{px}\}$.

(c) From the above characterisation of Θ by (3.8), (3.10) and (3.11) it is now clear that the non-zero $(p-1)$-form Θ is constructed from a set of linearly independent $(p-1)$ *spacelike* vectors, $\{Y_a \in Z^\perp : a = 1, \ldots, p-1\}$, which also belong to the (Lorentzian) tangent space $\mathcal{V}_x^p$ of a (timelike) p-brane through $x \in M$. Thus in $Z^\perp \subset T_x M^n$, the $(p-1)$-plane formed by $\{Y_a\}$ [and hence $\iota_Z \omega_x = (\sigma/\mu_x)\iota_Z \chi_x$] represents the 'spatial extension' of a p-brane through $x \in M$ with respect to an observer (x, Z) and it follows that

$$A_Z \equiv [\mathcal{G}(\iota_Z \chi_x, \iota_Z \chi_x)]^{\frac{1}{2}} > 0$$

is the spatial volume of the p-brane in $Z^\perp$. Now, projecting the foliating p-form χ onto $Z^\perp$ [as in (3.6)] and using $\mathcal{G}(\chi, \chi) = -\mu^2$ [(3.4)] we also have $-(\mu_x)^2 = -(A_Z)^2 + (Q_Z)^2$ - where

$$Q_Z \equiv [\mathcal{G}(\Pi_Z \chi_x, \Pi_Z \chi_x)]^{\frac{1}{2}} \geq 0$$

is the volume of the p-plane spanned by $\{V_\alpha\}$, when projected into $Z^\perp$. From the above relation connecting μ_x, A_Z and Q_Z we may define

$$\gamma(Z) \equiv (A_Z/\mu_x) = [1 - (Q_Z/A_Z)^2]^{-\frac{1}{2}}$$

and from (3.10) compute

$$
\begin{aligned}
[\mathcal{G}(\Theta, \Theta)]^{\frac{1}{2}} &= (\sigma/\mu_x) A_Z \\
&= \sigma \gamma(Z) \\
&= \sigma [1 - (Q_Z/A_Z)^2]^{-\frac{1}{2}}
\end{aligned}
$$

If an observer Z belongs to the tangent space $\mathcal{V}_x^p$ of a p-brane, then $\Pi_Z \omega_x = 0$, and hence $Q_Z = 0$ and $\gamma(Z) = 1$. In this case we have

$$[\mathcal{G}(\Theta, \Theta)]^{\frac{1}{2}}\bigg|_{Z \in \mathcal{V}_x^p} = \sigma$$

where $\Theta = \iota_Z \omega_x$. Thus $\mathcal{E}_Z \equiv [\mathcal{G}(\iota_Z \omega_x, \iota_Z \omega_x)]^{\frac{1}{2}}$ is indeed the energy per unit spatial volume of a p-brane with respect to any instantaneous observer (x, Z).

Now, given an instantaneous observer (x, Z), we can find a 'sufficiently small' neighborhood of $x \in M$ [where curvature tensor is negligible] which (by exponential map) can be regarded[6] as part of $T_x M^n$. In such a neighborhood of x, we compute the 'number density' with respect to (x, Z) by projecting $\mathbf{n}_x$ [(3.1)] into $\Lambda^{n-p}(Z^\perp)$ [as in (1.5)]:

$$\mathbf{n}_Z \equiv \Pi_Z [\star(\eta\omega)]_x = \eta_x \star [\tilde{Z} \wedge (-\iota_Z \omega_x)] \tag{3.12}$$

where we used the identity $\tilde{Z} \wedge \star\Psi = (-1)^{k-1} \star [\iota_Z \Psi]$ for $\Psi \in \Lambda^k(T_x M^n)$. Then, with respect to (x, Z), the number of p-branes of D intercepted by unit volume of an $(n-p)$-plane in $T_x M^n$ – orthogonal to the p-plane represented by the nonzero p-form $\tilde{Z} \wedge (-\iota_Z \omega_x)$ – is approximately given by [as in (1.6)]

$$\|\mathbf{n}_Z\| \equiv [\mathcal{G}(\mathbf{n}_Z, \mathbf{n}_Z)]^{\frac{1}{2}} = \eta_x [\mathcal{G}(\iota_Z \omega_x, \iota_Z \omega_x)]^{\frac{1}{2}} \tag{3.13}$$

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Finally, taking the product of $\|\mathbf{n}_Z\|$ in (3.13) and $\mathcal{E}_Z$ [(3.7)] we find the energy density U_Z (that (x, Z) measures) of the fluid $(\mathsf{D}, \eta, \sigma)$ locally characterised by ω :

$$U_Z = \eta_x \mathcal{G}(\iota_Z \omega_x, \iota_Z \omega_x) \geq 0 \qquad (3.14)$$

Stress Tensor for $(\mathsf{D}, \eta, \sigma)$:

Now U_Z is supposed to be equal to $\mathsf{T}(Z, Z)$ for every observer Z (see our discussion before Definition 1) for any given form of the stress tensor T. Then the structure of U_Z in (3.14) suggests the following definition of the *stress tensor* T for a multidimensional fluid $(\mathsf{D}, \eta, \sigma)$ with the local representation

$$\mathsf{T} = \eta \mathcal{G}(\iota_a \omega, \iota_b \omega) e^a \otimes e^b \qquad (3.15)$$

where $\{e^a\}$ are the local basis 1-forms (on M) dual to $\{X_b\}$ such that $e^a(X_b) = \delta^a{}_b$ for $a, b = 1, \ldots, n$ and $\iota_a \equiv \iota_{X_a}$. It is clear that T is symmetric, and for any observer (x, Z)

$$\begin{aligned} \mathsf{T}_x(Z, Z) &= \eta_x \mathcal{G}(\iota_a \omega_x, \iota_b \omega_x) e^a(Z) e^b(Z) \\ &= \eta_x \mathcal{G}(\iota_Z \omega_x, \iota_Z \omega_x) \geq 0 \end{aligned} \qquad (3.16)$$

Then by continuity $\mathsf{T}_x(W, W) \geq 0$ for all causal $W \in T_x M$ and hence T is a stress tensor. Furthermore, the equations (3.14) and (3.16) show that the energy density U_Z is equal to $\mathsf{T}_x(Z, Z)$ for every instantaneous observer (x, Z). Hence by the uniqueness property, the stress tensor for the fluid, $(\mathsf{D}, \eta, \sigma)$, is specified by the equation (3.15). We also remark that replacing ω in (3.15) by the energy-momentum 1-form, $\tilde{P}$, reproduces the stress tensor for the particle flows.

The stress tensor T defined in (3.15) for the fluid $(\mathsf{D}, \eta, \sigma)$ can be written in a form which is more suggestive as well as convenient for applications. From (3.4) we have

$$\begin{aligned} \det[g(V_\alpha, V_\beta)] &\equiv \mathcal{G}(\tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p, \ \tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p) \\ &\equiv \mathcal{G}(\chi, \chi) \equiv -\mu^2 \neq 0 \end{aligned} \qquad (3.17)$$

Now, expanding $\iota_a \omega$ in (3.15) in terms of χ^α [(3.9)],

$$\iota_a \omega = (\sigma/\mu) \sum_{\alpha=1}^{p} (-1)^{\alpha-1} g(V_\alpha, X_a) \chi^\alpha \qquad (3.18)$$

and inserting (3.18) in (3.15) we have

$$\begin{aligned} \mathsf{T} &= \eta(\sigma/\mu)^2 \sum_{\alpha,\beta} (-1)^{\alpha+\beta} \mathcal{G}(\chi^\alpha, \chi^\beta) g(V_\alpha, X_a) g(V_\beta, X_b) e^a \otimes e^b \\ &= \eta(\sigma/\mu)^2 \sum_{\alpha,\beta} C^{\alpha\beta} \tilde{V}_\alpha \otimes \tilde{V}_\beta \end{aligned} \qquad (3.19)$$

where $C^{\alpha\beta} \equiv (-1)^{\alpha+\beta}\mathcal{G}(\chi^\alpha, \chi^\beta)$ is the cofactor of the matrix element $\hat{g}_{\alpha\beta} \equiv [g(V_\alpha, V_\beta)]$ and from (3.17), the inverse of $\hat{g}_{\alpha\beta}$ is given by $\hat{g}^{\alpha\beta} \equiv C^{\alpha\beta}/(-\mu^2)$. From these definitions and (3.19), we have

$$\mathsf{T} = -(\sigma^2\eta)\hat{g}^{\alpha\beta}\tilde{V}_\alpha \otimes \tilde{V}_\beta \tag{3.20}$$

Defining

$$\hat{g} \equiv \hat{g}^{\alpha\beta}\tilde{V}_\alpha \otimes \tilde{V}_\beta \tag{3.21}$$

we note that $\hat{g}$ is simply a rank-2 symmetric tensor field on the Lorentzian manifold (M^n, g) and constructed only from the foliating vector fields $\{V_\alpha\}$. Now computing the components of $\hat{g}$ on a leaf L_p (whose tangent space is spanned by $\{V_\alpha\}$), we find from (3.21)

$$\hat{g}(V_\lambda, V_\nu) = \hat{g}_{\lambda\nu} \tag{3.22}$$

where we used the fact [see the definitions below (3.19)] that $\hat{g}^{\alpha\beta}$ is the inverse of the matrix element $\hat{g}_{\alpha\beta} \equiv g(V_\alpha, V_\beta)$. $\hat{g}^{\alpha\beta}$ exists since by (3.17) $\hat{g}$ is non-degenerate :

$$\det[g(V_\alpha, V_\beta)] \equiv \det[\hat{g}_{\alpha\beta}] = -\mu^2 \neq 0 \tag{3.23}$$

Then from (3.22) and (3.23), restriction of the symmetric tensor field $\hat{g}$ [(3.21)] onto each leaf L_p defines a metric on L_p - induced by the Lorentzian metric g on M. Since the leaves of the foliation are connected and timelike, $\hat{g}$ - restricted to a leaf - is also a Lorentzian metric of constant index. Thus we have a simple interpretation of (3.20) that T is proportional to a metric $\hat{g}$ on each integral submanifolds, and from (3.20)-(3.21)

$$\mathsf{T} = -\rho\hat{g} \tag{3.24}$$

where the *positive* function (on M), $\rho \equiv \sigma^2\eta$, is the energy density measured by all observers tangential to the leaves L_p.

As we mentioned earlier [see remark(2) below **Definition 3** in **section 2**], a p-foliation of an n-dimensional manifold may also be prescribed by $(n-p) \equiv q$ constraint 1-forms $\{\theta^i\}$ where $\theta^i(V_\alpha) = 0$ and by suitable linear combinations from the linearly independent set $\{\tilde{\theta}^i\}$ we can get an orthonormal set of q normal fields $\{N_k\}$ such that

$$\begin{aligned}
g(N_i, N_j) &= \delta_{ij} \\
g(N_i, V_\alpha) &= 0
\end{aligned}$$

Then the sets $\{V_\alpha\}$ and $\{N_k\}$ together form a local basis for the tangent spaces of M and $\{\tilde{N}_k\}$ are the new constraint 1-forms satisfying the integrability condition (2.2). In terms of

these normal fields (3.24) can be written as

$$\mathsf{T} = -\rho(g - \sum_{k=1}^{q} \tilde{N}_k \otimes \tilde{N}_k) \quad ; \quad q \equiv (n - p) \tag{3.25}$$

4 Dynamics and Symmetries of Multidimensional Fluids

The stress tensor, T [in equivalent forms (3.15), (3.24), (3.25)], associated with a p-dimensional fluid is simply a symmetric tensor field on (M^n, g). However, if the spacetime admits Killing vector fields, K, then T may acquire new symmetries through the Einstein field equation

$$\mathsf{G} = \mathsf{T} \tag{4.1}$$

where $\mathsf{G} \equiv \mathsf{Ric} - \frac{1}{2}g\mathsf{R}$ is the Einstein tensor of the spacetime (M^n, g), constructed from the Ricci tensor Ric and the scalar curvature R. Since Killing vector fields generate local isometries of (M, g), we have[7] $\mathsf{L_K Ric} = 0 = \mathsf{L_K R}$ where $\mathsf{L_K}$ is the Lie derivation with respect to K. Then from the definition of the Einstein tensor G it also follows $\mathsf{L_K G} = 0$, and hence by (4.1), $\mathsf{L_K T} = 0$.

<u>Proposition 1</u>: If K is a Killing vector field and $(\mathsf{D}, \eta, \sigma)$ is a p-dimensional fluid on (M^n, g), then the world density function η satisfies $\mathsf{L_K}\eta = 0$.

<u>Proof</u>: Taking trace of both sides of (4.1) with T given by (3.25), we find

$$\rho = (\frac{1}{p})(\frac{n}{2} - 1)\mathsf{R} \tag{4.2}$$

where n and p are the dimensions of M and the distribution D, respectively. Since $\mathsf{L_K R} = 0$, and $\rho \equiv \sigma^2 \eta$, it follows from (4.2) that $\mathsf{L_K}\eta = 0$. $\square$

<u>Corollary 1</u>: If T in (4.1) is the fluid stress tensor $\mathsf{T} = -\rho\hat{g}$ [(3.24)], and K is a Killing vector field on (M, g), then $\mathsf{L_K}\hat{g} = 0$.

<u>Proof</u>: Since $\mathsf{L_K T} = 0$, it follows that $-(\mathsf{L_K}\rho)\hat{g} - \rho(\mathsf{L_K}\hat{g}) = 0$. Then the corollary follows from the fact that $\mathsf{L_K}\rho = 0$, by Proposition 1, and $\rho \neq 0$. $\square$

We now discuss the significance of Corollary 1 which suggests that the symmetries of a spacetime (M^n, g) are also the symmetries of the leaves $\{L_p\}$ of a given foliation of M. First, we prove a consequence of $\mathsf{L_K}\hat{g} = 0$, where $\hat{g} = g - \sum_{k=1}^{q} \tilde{N}_k \otimes \tilde{N}_k$ [(3.25)], the set $\{N_k\}$ is normal to the foliating vector fields $\{V_\alpha\}_{\alpha=1}^{p}$, $g(N_i, N_j) = \delta_{ij}$ and $q \equiv (n - p)$.

<u>Lemma 1</u>: If $\mathsf{L}_\mathsf{K}\hat{g} = 0$ where K is Killing, then $\mathsf{L}_\mathsf{K}N_j = \sum_{k \neq j} a^k N_k$ where the a^k's are real numbers.

<u>Proof</u>: Taking the Lie derivative of $\hat{g}$ we have $\mathsf{L}_\mathsf{K}(g - \sum_{i=1}^q \tilde{N}_i \otimes \tilde{N}_i) = 0$. Since K is Killing (and hence $\mathsf{L}_\mathsf{K}g = 0$),

$$\sum_{i=1}^q (\mathsf{L}_\mathsf{K}\tilde{N}_i \otimes \tilde{N}_i + \tilde{N}_i \otimes \mathsf{L}_\mathsf{K}\tilde{N}_i) = 0 \tag{4.3}$$

Now, evaluating the symmetric tensor in (4.3) on $\{V_\alpha, N_j\}$

$$0 = \sum_{i=1}^q g(\mathsf{L}_\mathsf{K}N_i, V_\alpha)\delta_{ij} = g(\mathsf{L}_\mathsf{K}N_j, V_\alpha) \quad \forall j, \alpha$$

Similarly, evaluating (4.3) on $\{N_j, N_l\}$ we find

$$\begin{aligned} 0 &= \sum_{i=1}^q g(\mathsf{L}_\mathsf{K}N_i, N_j)\delta_{il} + \sum_{i=1}^q g(\mathsf{L}_\mathsf{K}N_i, N_l)\delta_{ij} \\ &= g(\mathsf{L}_\mathsf{K}N_l, N_j) + g(\mathsf{L}_\mathsf{K}N_j, N_l) \end{aligned}$$

Now, substituting $i = j = l$, we have $g(\mathsf{L}_\mathsf{K}, N_j, N_i) \quad \forall i$. Thus, $\mathsf{L}_\mathsf{K}N_j$ is normal to N_j as well as $V_\alpha \quad \forall \alpha$, and hence, expanding $\mathsf{L}_\mathsf{K}N_j$ in the basis $\{V_1, \ldots, V_p, N_1, \ldots, N_q\}$, it follows that $\mathsf{L}_\mathsf{K}N_j = \sum_{k \neq j} a^k N_k$. $\square$

<u>Proposition 2</u>: If K is a Killing vector field on (M, g), then $\mathsf{L}_\mathsf{K}\omega = 0$, where $\omega \equiv \sigma(\tilde{V}_1 \wedge \ldots \wedge \tilde{V}_p)/\mu$ - defined by the foliating vector fields $\{V_\alpha\}$ - is the local representation of the distribution D characterising a p-dimensional fluid $(\mathsf{D}, \eta, \sigma)$ in a spacetime (M^n, g), and $-\mu^2 \equiv \det[g(V_\alpha, V_\beta)]$.

<u>Proof</u>: In terms of the spacelike orthonormal set $\{N_k\}$

$$\omega = \sigma(-1)^{1+pq} \star (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q) \tag{4.4}$$

where $\star$ is the Hodge operator induced by the spacetime metric g and $g(N_k, V_\alpha) = 0 \; \forall k = 1, \ldots, q \; ; \; \forall \alpha = 1, \ldots, p$. Since for any Killing field K, L_K commutes with $\star$ and metric dual operation

$$\begin{aligned} \mathsf{L}_\mathsf{K} \star (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q) &= \star[\mathsf{L}_\mathsf{K}(\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q)] \\ &= \star[\sum_{j=1}^q (-1)^{j-1}\widetilde{\mathsf{L}_\mathsf{K}N_j} \wedge \Pi_j] \end{aligned}$$

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where $\Pi_j = \tilde{N}_1 \wedge \ldots \wedge \tilde{N}_{j-1} \wedge \tilde{N}_{j+1} \ldots \wedge \tilde{N}_q$. Now, using $\mathsf{L}_\mathsf{K} N_j = \sum_{k \neq j} a^k N_k$ from Lemma 1, it follows that each term in the above expansion vanishes. Hence $\mathsf{L}_\mathsf{K} \star (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q) = 0$, and by Lie derivation of (4.4) with respect to K we conclude that $\mathsf{L}_\mathsf{K} \omega = 0$. $\square$

Now we derive the dynamical consequences of (4.1) with the foliating stress tensor [(3.25)]. First, we recall that the Einstein tensor G is divergence-free, $\nabla \cdot \mathsf{G} = 0$, and hence for any stress tensor (4.1) implies $\nabla \cdot \mathsf{T} = 0$. In our case of interest, T is the stress tensor for a p-foliation, given by [(3.25)] $\mathsf{T} = -\rho(g - \sum_{k=1}^q \tilde{N}_k \otimes \tilde{N}_k)$. Then, we have

$$\nabla \cdot \tilde{\mathsf{T}} = -\tilde{d}\rho + \sum_{k=1}^q \{\nabla \cdot (\rho N_k)\} N_k + \sum_{k=1}^q \rho \nabla_{N_k} N_k = 0 \tag{4.5}$$

__Proposition 3__: For each integral manifold (p-brane) of the distribution, D, characterising a p-dimensional fluid $(\mathsf{D}, \eta, \sigma)$ in (M^n, g), the mean curvature field $\mathsf{H} = 0$.

__Proof__: The proposition involves a local assertion. Since the distribution, D, is integrable, for every point $x \in M^n$ there exists an integral manifold, $\mathcal{S}^p$, passing through x, and (by Frobenius Theorem) there is an open neighborhood of x, $\mathcal{U} \subset M^n$, where we may choose an orthonormal moving frame $X_1, \ldots, X_p, N_1, \ldots, N_q$ such that $\{X_a\}$ are tangent to $\mathcal{S}^p$ and $\{N_j\}$ are normal to $\mathcal{S}^p$. Since $\mathcal{S}^p$ is timelike, $g(X_a, X_a) = \epsilon_a = \pm 1$. Then the mean cuvature[11] vector field H of $\mathcal{S}^p \subset M^n$ has the following local representation:

$$\mathsf{H} = \sum_{j=1}^q \sum_{a=1}^p \epsilon_a g(\nabla_{X_a} X_a, \ N_j) N_j \tag{4.6}$$

From (4.5), $g(\nabla \cdot \tilde{\mathsf{T}}, N_j) = 0$, which implies

$$-N_j(\rho) + \nabla \cdot (\rho N_j) + \rho \sum_{k=1}^q g(\nabla_{N_k} N_k, N_j) = 0 \tag{4.7}$$

Now, inserting the following expansion

$$\begin{aligned}
\nabla \cdot (\rho N_j) &= N_j(\rho) + \rho \nabla \cdot N_j \\
&= N_j(\rho) + \rho \sum_{a=1}^p \epsilon_a g(\nabla_{X_a} N_j, \ X_a) + \rho \sum_{k=1}^q g(\nabla_{N_k} N_j, \ N_k) \\
&= N_j(\rho) - \rho \sum_{a=1}^p \epsilon_a g(\nabla_{X_a} X_a, \ N_j) - \rho \sum_{k=1}^q g(\nabla_{N_k} N_k, \ N_j)
\end{aligned}$$

in the equation (4.7) we have $\sum_{a=1}^p \epsilon_a g(\nabla_{X_a} X_a, \ N_j) = 0$. Then, by (4.6), it follows that $g(\mathsf{H}, N_j) = 0$ for $j = 1, \ldots, q$. Hence $\mathsf{H} = 0$. $\square$

Remark: By the above proposition, the equation of motion of a p-brane, $\mathcal{S}$, in a multidimensional fluid $(\mathsf{D}, \eta, \sigma)$ is given by $\mathsf{H} = 0$. Also, it can be shown[13] that H and ω (the local representation of D) are related by $\iota_N d\omega|_{\mathcal{S}} = -g(\mathsf{H}, N)|_{\mathcal{S}}$, for every vector field, N, normal to the p-brane. Then $\mathsf{H} = 0$ implies $\iota_N d\omega|_{\mathcal{S}} = 0$.

In order to derive a further dynamical consequence of (4.5), we need the following

<u>Lemma 2</u>: The world-density function, η, satisfies $\iota_V (d\eta + \eta\lambda) = 0$, where V is any vector field tangent to the integral submanifolds of the fluid, $(\mathsf{D}, \eta, \sigma)$ and λ is some 1-form.

<u>Proof</u>: From (4.5), $g(\nabla \cdot \tilde{\mathsf{T}}, V) = 0$, which implies

$$
\begin{aligned}
V(\rho) &= \rho \sum_{k=1}^{q} g(\nabla_{N_k} N_k, V) \\
&= \rho \sum_{k=1}^{q} d\tilde{N}_k(N_k, V) \\
&= -\rho \sum_{k=1}^{q} \lambda^k{}_k(V)
\end{aligned}
$$

where we have used the identity $\nabla_{N_k} \tilde{N}_k = \iota_{N_k} d\tilde{N}_k$ and the fact that the constraint 1-forms $\{\tilde{N}_k\}$ describing the p-foliation must satisfy the integrability conditions [(2.2)] $d\tilde{N}_k = \sum_{j=1}^{q} \lambda^j{}_k \wedge \tilde{N}_j$, λ^j_k being suitable 1-forms. Since $\rho \equiv \sigma^2 \eta$, defining $\lambda \equiv \sum_{k=1}^{q} \lambda^k_k$, we have $[d\eta + \eta\lambda](V) = 0$. $\square$

<u>Proposition 4</u> : The world density function η on (M, g) satisfies $d \star (\eta\omega) = 0$.

Proof : From the expression for the foliation p-form ω [(4.4)]

$$
\begin{aligned}
d \star (\eta\omega) &= \sigma(-1)^{1+pq}\{\eta d(\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q) \\
&\quad + d\eta \wedge (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q)\}
\end{aligned}
$$

Using the integrability conditions (2.2) for $\{\tilde{N}_k\}$ we compute

$$
\begin{aligned}
d(\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q) &= \sum_{k=1}^{q} (-1)^{k-1} d\tilde{N}_k \wedge \Phi^k \\
&= (\sum_{k=1}^{q} \lambda^k{}_k) \wedge (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q)
\end{aligned}
$$

where $\Phi^k \equiv \tilde{N}_1 \wedge \ldots \wedge \tilde{N}_{k-1} \wedge \tilde{N}_{k+1} \ldots \wedge \tilde{N}_q$. Then we have

$$
d \star (\eta\omega) = \sigma(-1)^{1+pq}(d\eta + \eta \sum_{k=1}^{q} \lambda^k{}_k) \wedge (\tilde{N}_1 \wedge \ldots \wedge \tilde{N}_q)
$$

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Since the 1-form $(d\eta + \eta \sum \lambda^k{}_k)$ does not have any tangential components (by lemma 2) and its $\{\tilde{N}_k\}$-components do not contribute in the above equation, $d \star (\eta\omega) = 0$. $\square$

<u>Remark</u>: Using the coderivative operator $\delta = (-1)^{n(k+1)} \star d\star$ on differential k-forms in a Lorentzian manifold (M^n, g), we can write $d \star (\eta\omega) = 0$ as $\delta(\eta\omega) = 0$.

Thus our program - of investigating a p-dimensional fluids as a source of spacetime curvature - would be to solve the Einstien equations (4.1) with the stress tensor T in (3.15), and to specify g and (ω, η) – the local representation of the fluid. In the next section we shall work out a complete solution for a fluid characterised by a 2-dimensional distribution on a spacetime (M^4, g).

5 Spherically Symmetric 2-Foliation

As an application of our results in the previous sections we consider a 2-foliation (due to string world-sheets) of a static spherically symmetric spacetime (M^4, g) where, in the local chart (t, r, θ, ϕ), metric g is of the form

$$g = -h^2(r)dt \otimes dt + f^2(r)dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi) \tag{5.1}$$

Using an orthonormal basis (5.1) can be written as

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 \tag{5.2}$$

where coframes are

$$e^0 = h(r)dt; \quad e^1 = f(r)dr; \quad e^2 = rd\theta; \quad e^3 = r\sin\theta d\phi \tag{5.3}$$

with the dual basis given by

$$X_0 = (1/h)\partial_t; \quad X_1 = (1/f)\partial_r; \quad X_2 = (1/r)\partial_\theta; \quad X_3 = (1/r\sin\theta)\partial_\phi \tag{5.4}$$

such that $e^a(X_b) = \delta^a{}_b$. It is clear from (5.1) that (M^4, g) has four Killing vector fields :

$$
\begin{aligned}
\mathsf{K}_0 &= \partial_t \\
\mathsf{K}_1 &= \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \\
\mathsf{K}_2 &= -\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi \\
\mathsf{K}_3 &= \partial_\phi
\end{aligned}
\tag{5.5}
$$

<u>Foliation 2-Form</u>: If the spacetime specified by (5.1) is to be foliated by string world-sheets, we must find the appropriate 2-form $\omega \in \Lambda^2(M)$ satisfying [Proposition 2]

$$\mathsf{L}_{\mathsf{K}_i}\omega = 0 \tag{5.6}$$

where the Killing vector fields $\{\mathsf{K}_i\}$ are given in (5.5). The most general 2-form satisfying (5.6), on the chosen spacetime[(5.1)], must be

$$\omega = c_1(r)dt \wedge dr + c_2(r)\sin\theta d\theta \wedge d\phi \tag{5.7}$$

Locally, ω is required to be decomposable, and for timelike foliation ω must satisfy $\mathcal{G}(\omega, \omega) < 0$. These two constraints together with (5.7) uniquely (up to a scalar function) specify the structure of the foliation 2-form so that

$$\omega = c_1(r)dt \wedge dr \tag{5.8}$$

Without any loss of generality we can normalise (5.8) by $\mathcal{G}(\omega, \omega) = -1$, and the foliation 2-form is then given by

$$\omega = h(r)f(r)dt \wedge dr = e^0 \wedge e^1 \tag{5.9}$$

where we used the equations $(5.1) - (5.4)$. It is now easy to see that the contraint form $\star\omega$ satisfies the integrability condition $[(2.3)]$:

$$d \star \omega = -d(e^2 \wedge e^3) = (2/r)dr \wedge \star\omega \tag{5.10}$$

Thus ω, indeed, determines a 2-foliation. Now, introducing string rest-mass per unit length, σ, the foliation 2-form for the extended particle flow (ω, η) is written as

$$\omega = \sigma e^0 \wedge e^1 \tag{5.11}$$

<u>Stress Tensor</u>: From (3.15) and (5.11), the associated stress tensor is given by

$$
\begin{aligned}
\mathsf{T} &= \eta\mathcal{G}(\iota_a\omega, \iota_b\omega)e^a \otimes e^b \\
&= \eta\sigma^2\{\mathcal{G}(e^1, e^1)e^0 \otimes e^0 + \mathcal{G}(-e^0, -e^0)e^1 \otimes e^1\} \\
&= \eta\sigma^2\{e^0 \otimes e^0 - e^1 \otimes e^1\}
\end{aligned} \tag{5.12}
$$

where T is expanded in the orthonormal basis given in (5.3) and (5.4). The density function, η, can be obtained from Proposition 4 and (5.11):

$$0 = d \star (\eta\omega) = -\sigma d(\eta r^2) \wedge \sin\theta d\theta \wedge d\phi \tag{5.13}$$

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Then (5.13) implies $\partial_r(\eta r^2) = 0$ and hence ηr^2 is constant. Thus the density function for 2-foliation is given by

$$\eta = c/r^2 \tag{5.14}$$

where c is some positive constant since η is defined to be positive.

Solution to Einstein's Equations: For complete specification of the string-field flow we must find the functions $h(r)$ and $f(r)$ from the Einstein equation (4.1) with the stress tensor in (5.12). For convenience (4.1) is written in the following form :

$$\mathsf{P}_a = \mathsf{T}_{ab}e^b - (\Gamma/2)g_{ab}e^b \tag{5.15}$$

where $\mathsf{P}_a \equiv \mathsf{Ric}(X_a, X_b)e^b$ are the Ricci 1-forms, and $\Gamma \equiv trace\mathsf{T} = -2\eta\sigma^2$ by (5.12). Now, computing the Ricci forms with respect to an orthonormal basis[(5.3), (5.4)], we find

$$
\begin{aligned}
\mathsf{P}_0 &= (1/f^2)[(h''/h) - (h'/h)(f'/f) + (2/r)(h'/h)]e^0 \\
\mathsf{P}_1 &= -(1/f^2)[(h''/h) - (h'/h)(f'/f) - (2/r)(f'/f)]e^1 \\
\mathsf{P}_2 &= [(1/rf^2)\{-(h'/h) + (f'/f)\} + (1/r^2)\{1 - (1/f^2)\}]e^2 \\
\mathsf{P}_3 &= [(1/rf^2)\{-(h'/h) + (f'/f)\} + (1/r^2)\{1 - (1/f^2)\}]e^3
\end{aligned}
\tag{5.16}
$$

From (5.2), (5.12), (5.15) and defining $\rho \equiv \eta\sigma^2$, we also have

$$\mathsf{P}_0 = 0; \quad \mathsf{P}_1 = 0; \quad \mathsf{P}_2 = \rho e^2; \quad \mathsf{P}_3 = \rho e^3 \tag{5.17}$$

Then (5.16) and (5.17) imply

$$(h''/h) - (h'/h)(f'/f) + (2/r)(h'/h) = 0 \tag{5.18}$$

$$(h''/h) - (h'/h)(f'/f) - (2/r)(f'/f) = 0 \tag{5.19}$$

$$(1/rf^2)\{-(h'/h) + (f'/f)\} + (1/r^2)\{1 - (1/f^2)\} = \rho \tag{5.20}$$

To solve these equations, first, we note that subtracting (5.19) from (5.18) gives

$$(h'/h) + (f'/f) = 0 \tag{5.21}$$

Integrating (5.21) and choosing the integration constant to be 0, we have

$$hf = 1 \tag{5.22}$$

Now inserting (5.21) in (5.18) and defining $a(r) \equiv h^2(r)$ we get

$$a'' + (2/r)a' = 0 \tag{5.23}$$

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The general solution to (5.23) is found to be

$$a(r) \equiv h^2(r) = (\beta - 2m/r) \tag{5.24}$$

where β and m are constants with $m > 0$. Then using (5.21), (5.22) and (5.24) in the equation (5.20) we find

$$(1/r^2)(1 - \beta) = \rho \tag{5.25}$$

It is clear from (5.25) that

$$\beta \neq 1 \Leftrightarrow \rho \neq 0 \tag{5.26}$$

Hence for non-vanishing string field flow (ω, η), the constant β can not be equal to 1. Comparing (5.25) with (5.14) and using $\rho \equiv \eta\sigma^2$ we also have

$$\beta = 1 - c\sigma^2 \tag{5.27}$$

Collecting our results in (5.22), (5.24) and (5.27) the spacetime metric [(5.1)] is now given by

$$\begin{aligned} g &= -(1 - c\sigma^2 - 2m/r)dt \otimes dt + (1 - c\sigma^2 - 2m/r)^{-1}dr \otimes dr \\ &\quad +r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi) \end{aligned} \tag{5.28}$$

and from (5.22) the foliating 2-form $\omega[(5.11)]$ takes the following form :

$$\omega = \sigma dt \wedge dr \tag{5.29}$$

Thus the equations (5.28), (5.29) and (5.14) completely determine the local flow of a 2-dimensional fluid generated by radial strings in a static spherically symmetric spacetime. The metric in (5.28) may be interpreted as representing a spacetime associated with a particle of mass m (at $r = 0$) surrounded by spherically symmetric distribution of strings with density $\eta = c/r^2$ [(5.14)].

Properties of the Solution (5.28) :

(a) For $c\sigma^2 < 1$, this solution has a horizon of radius

$$r_0 = 2m/(1 - c\sigma^2) \tag{5.30}$$

The equation (5.30) shows that the Schwarzschild radius for the mass m is enhanced by a factor $(1 - c\sigma^2)^{-1} > 1$.

(b) For $m = 0$, there is no horizon but the spacetime has a **naked** singularity at $r = 0$. To see this, we substitute $p = 2$ (dimension of foliation) and $n = 4$ (spacetime dimension) in (4.2) to find the Ricci (scalar) curvature :

$$\mathsf{R} = 2c\sigma^2/r^2 \tag{5.31}$$

We also compute an invariant scalar constructed from the curvature 2-forms R_{ab}, where $a, b = 0, 1, 2, 3$:

$$\star(\mathsf{R}^{ab} \wedge \star\mathsf{R}_{ab}) = 24m^2/r^6 + 8m(c\sigma^2)/r^5 + 2(c^2\sigma^4)/r^4 \tag{5.32}$$

Besides the existence of the singularity at $r = 0$, (5.31) and (5.32) also imply that the spacetime remains curved with $m = 0$ - that is - with the foliating strings alone, and the metric, (5.28), remains well-behaved even with $m = 0$ and $c\sigma^2 = 1$.

(c) Using (5.2), if we rewrite the foliation stress tensor, (5.12), in the form, $\mathsf{T} = -\rho(g - e^2 \otimes e^2 - e^3 \otimes e^3)$, then for any causal vector V we have

$$\begin{aligned}
\mathsf{T}(V, V) &= -\rho\{g(V, V) - [g(X_2, V)]^2 - [g(X_3, V)]^2\} \\
&\geq -\rho g(V, V) = (1/2)(trace\,\mathsf{T})g(V, V)
\end{aligned} \tag{5.33}$$

where we have used the inequalities $\rho \geq 0$ and $g(V, V) \leq 0$. The equation (5.33) shows that the stress tensor (5.12) satisfies the *strong energy condition*, and this also means that the gravitational field - generated by the foliating string world-sheets - is attractive.

Thus the fluid of string world-sheets gives rise to a non-trivial solution (to Einstein's equations) - which is non-flat, static and spherically symmetric with a naked singularity. Such a network of line-like objects could be used to model a multilayer star where the constituents of each layer follows different equation of state.

Modelling Star with 2-Foliation :

Following a suggestion in reference [12], we consider a two-layer star in which the **core** consists of a (spatially isotropic) perfect fluid, and the exterior is formed by a spherically symmetric distribution of strings as in (5.29). Thus the stress tensor for the core is given by

$$\mathsf{T}_c = (\rho_c + \nu)\tilde{u} \otimes \tilde{u} + \nu g_c \tag{5.34}$$

where u is a unit timelike vector field - called flow vector field, ρ_c is the density of the core and taken to be a constant, and ν is the spatially isotropic pressure function. Then,

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in the chart (t, r, θ, ϕ), the metric tensor g_c for the core of radius r_c is given by a special case ($\rho_c \equiv constant$) of the well-known Oppenheimer-Volkov[14] solution to the Einstein field equations :

$$
\begin{aligned}
g_c &= -(1/4)[3(1 - \frac{1}{3}\rho_c r_c^2)^{\frac{1}{2}} - (1 - \frac{1}{3}\rho_c r^2)^{\frac{1}{2}}]^2 dt \otimes dt \\
&\quad + (1 - \frac{1}{3}\rho_c r^2)^{-1} dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)
\end{aligned}
\tag{5.35}
$$

where $r \in (0, r_c)$ and $u = \partial_t$ in (5.34). Futhermore, the isotropic pressure $\nu(r)$ can be obtained from the Oppenheimer-Volkov[14] equation :

$$
(3\nu + \rho_c)^2/(\nu + \rho_c)^2 = [(3\nu_0 + \rho_c)^2/(\nu_0 + \rho_c)^2](1 - \frac{1}{3}\rho_c r^2)
\tag{5.36}
$$

where ν_0 is the pressure at $r = 0$.

The metric for the spacetime region foliated by the radial strings [(5.29)] is taken as [(5.28)] :

$$
\begin{aligned}
g_s &= -(1 - q - 2m/r)dt \otimes dt + (1 - q - 2m/r)^{-1} dr \otimes dr \\
&\quad + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)
\end{aligned}
\tag{5.37}
$$

where $q \equiv c\sigma^2$ [from (5.28)] is a positive constant. In (5.37) we require $r \in (r_c, r_s)$ with r_s as star-radius, and $r_s > r_c$. For $r > r_s$, we have the Schwarzschild vacuum metric :

$$
\begin{aligned}
g_v &= -(1 - 2m_v/r)dt \otimes dt + (1 - 2m_v/r)^{-1} dr \otimes dr \\
&\quad + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)
\end{aligned}
\tag{5.38}
$$

To complete our model we need to match these metrics continuously across the boundaries of different layers. The matching condition for g_s and g_c at $r = r_c$ is given by $g_s\big|_{r_c} = g_c\big|_{r_c}$, and hence from (5.35) and (5.37) we have

$$
2m = r_c(\frac{1}{3}\rho_c r_c^2 - q)
\tag{5.39}
$$

Similarly, from (5.37) and (5.38), the continuity of g_s and g_v at $r = r_s$ implies

$$
2m_v = 2m + qr_s = \frac{1}{3}\rho_c r_c^3 + q(r_s - r_c)
\tag{5.40}
$$

where $r_s > r_c$. Now, if we impose the condition [see (5.35)]

$$
\frac{1}{3}\rho_c r_c^2 < 1
\tag{5.41}
$$

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then it follows from (5.39), (5.40) and (5.41) that $q < 1$, and

$$r_c > 2m/(1-q) \quad ; \quad r_s > 2m_v \tag{5.42}$$

The above conditions, (5.41)-(5.42), ensure that the spacetime regions specified by the corresponding metrics [(5.35), (5.37) and (5.38)] are static and free from any singularities. We also remark that (5.41) naturally follows from the condition that prohibits gravitational collapse of the core. To see this, first we note that the pressure $\nu = 0$ at the core-surface $r = r_c$, and this implies [from (5.36)]

$$1 = [(3\nu_0 + \rho_c)^2/(\nu_0 + \rho_c)^2](1 - \frac{1}{3}\rho_c r_c^2) \tag{5.43}$$

From the above equation (5.43), we find

$$\frac{1}{3}\rho_c r_c^2 = 4\nu_0(2\nu_0 + \rho_c)/(3\nu_0 + \rho_c)^2 \tag{5.44}$$

The right side of (5.44) can be easily seen to be an increasing function of the central pressure ν_0. However, evaluating the limit of (5.44) as $\nu_0 \to \infty$, we get

$$\frac{1}{3}\rho_c r_c^2\Big|_{\nu_0 \to \infty} = \frac{8}{9} \tag{5.45}$$

The equation (5.45) shows that there exists a maximum $R_m \equiv r_c\big|_{\nu_0 \to \infty}$ for the core-radius r_c with the given density ρ_c, and hence

$$\frac{1}{3}\rho_c r_c^2 \leq \frac{1}{3}\rho_c R_m^2 < 1 \tag{5.46}$$

The above inequality provides the validity of the condition (5.41) which leads to the model of a non-collapsing star with spherically symmetric distribution of strings.

Our discussion on 2-foliation suggests that a complete general-relativistic theory of multidimensional perfect fluids may enable us to investigate a new class of collapse problems, the possible formation of horizons and the nature of the associated singularities.

6 Conclusion :

We emphasize that the results in this paper strictly follow from the concepts introduced in **Definition 4** which offer a precise description of p-dimensional fluids. Consequently, the foliation p-form ω [with $\mathcal{G}(\omega, \omega) < 0$] together with the *uniqueness* property of stress-energy tensors directly lead to our formulation of the dynamics and symmetries of such

self-gravitating systems. Then the local decomposable p-form ω which defines a timelike p-foliation of a spacetime and the *world density* function η, enable us to introduce the local number-density [(3.1)] of the leaves of the foliation. They also give rise to the concept of spatial volume (and energy) of a p-brane with respect to any *observer* [remarks (a), (b) and (c) after (3.7)]. These two ingredients then naturally motivates a precise definition of the stress-energy tensor T[(3.15)] for multidimensional perfect fluids.

It is interesting to observe, from the equation [(4.2)] relating trace(T) and the the the scalar curvature R, that a 2-dimensional spacetime ($n = 2$) does not admit any *massive* particle ($p = 1$) flow, however, only massless flows are consistent with the field equations. In fact, a slight modification of our Definition 4 permits construction of null foliations, where ω satisfies $\mathcal{G}(\omega, \omega) = 0$.

Furthermore, ω provides the local description of the dynamics [Proposition 3 and Proposition 4] of p-branes and carries the symmetries [Proposition 1 and Proposition 2] of a foliated spacetime, and we have demonstrated, in section 5, that the spacetime symmetries determine to a large extent the structure of foliations [(5.7)] and hence the associated stress tensor. This example suggests the posssibility of other non-trivial solutions to the Einstein equations in a spacetime region dominated by extended objects like strings and membranes.

Appendix : Definition of $\mathcal{G}$

Given a semi-Riemannian manifold (M^n, g), the symmetric bilinear form $\mathcal{G}$ on the vector space $\Lambda^p(M)$ of differential p-forms on M is, first, defined[9] on decomposable p-forms and then the definition is extended to any p-forms by linearity. Consider two decomposable p-forms Θ and Φ given by

$$\Theta = \alpha_1 \wedge \ldots \wedge \alpha_p \quad ; \quad \Phi = \beta_1 \wedge \ldots \wedge \beta_p$$

where $\alpha_i, \ \beta_i \in \Lambda^1(M)$. Then, by definition, $\mathcal{G}(\Theta, \Phi) \equiv \det[g(\tilde{\alpha}_i, \tilde{\beta}_j)]$ – where $\tilde{\alpha}$ is the metric dual of the 1-form α. Now, for any pair of p-forms ω and χ, $\mathcal{G}$ has the following useful properties :

$$\omega \wedge \star\chi = \mathcal{G}(\omega, \chi) \star 1 \quad ; \quad \mathcal{G}(\star\omega, \star\chi) = (-1)^s \mathcal{G}(\omega, \chi)$$

where $\star 1$ is the volume form on M induced by the metric g, and s is the index of g.

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